Static Bifurcations in Electric Power Networks: Loss of Steady-State Stability and Voltage Collapse

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Abstract —This paper presents an analysis of static stability in electric power systems. The study is based on a model consisting of the classical swing equation characterization for generators and constant admittance, *PV* bus and/or *PQ* bus load representations which, in general, leads to a semi-explicit (or constrained) system of differential equations. A precise definition of static stability is given and basic concepts of static bifurcation theory are used to show that this definition does include conventional notions of steady-state stability and voltage collapse, but it provides a basis for rigorous analysis. Static bifurcations of the load flow equations are analyzed using the Liapunov-Schmidt reduction and Taylor series expansion of the resulting reduced bifurcation equation. These procedures have been implemented using symbolic computation (in MASYMA). It is shown that static bifurcations of the load flow equations are associated with either divergence-type instability or loss of causality. Causality issues are found to be an important factor in understanding voltage collapse and play a central role in organizing global power system dynamics when loads other than constant admittance are present.

I. Introduction

PROBLEMS associated with the steady-state stability and voltage collapse of electric power systems have become increasingly important during the past decade and have consequently received the attention of many investigators. To a large extent, this is due to the fact that the present-day power system operating environment substantially increases the difficulty of maintaining acceptable system voltage profiles. Low voltage can result in loss of stability and voltage collapse and ultimately to cascading power outages. Indeed, voltage difficulties have been associated with major incidents in Italy, France, Britain, Japan, and the U.S. (see [1]–[9] for a sample of the literature). Several factors have contributed to this situation, including the adoption of higher transmission voltages, the relative decrease in the reactive power output of large generating units, and the shift in power flow patterns associated with changing fuel costs and generation availability.

The severity and extended time duration of recent major system disturbances are responsible for the current interest in comprehensive system restoration plans (Kafka *et al.* [10], Blankenship and Trygar [11]). Here again, steady-state stability and voltage collapse are vital issues in the reconstruction of a power system in minimum time following a major system failure. Such incidents, although very rare, are of serious concern to the electric utility industry, which is actively pursuing the development of computer-aided procedures for minimizing the effect of such eventualities.

In response to these concerns, there continues to be a substantial international effort to develop on-line voltage control algorithms based on some form of optimal (active and reactive) power flow formulation (Hano, *et al.* [12], Aldrich *et al.* [13], Burchet *et al.* [14]). This is a very difficult problem and even the structure of an appropriate performance index is not obvious (the discussions in Savulesco [15] and Capasso *et al.* [16] are interesting in this regard). In fact, it is likely that a multiple objective formulation will prove necessary. The experience in France and Italy suggests that a practical control algorithm will identify critical buses and maintain tight control of voltage on those buses. But how does one identify the critical buses? In our view, this and other basic questions require answers if comprehensive, on-line voltage control is to become a reality.

It is common to associate steady-state stability with the ability of the transmission network to transport real power (for example, see the discussion of transmission capacity in [32]) and to associate voltage collapse with the inability to provide reactive power at the necessary locations within the network as described by Lachs [4], [34]. As meaningful as these interpretations may be in appropriate circumstances, a clear understanding of steady-state stability and voltage collapse can only be achieved by considering them both as arising from a common, well-defined origin. The main purpose of this paper is to establish the framework for such a point of view. Indeed, there do not now exist precise definitions of steady-state stability or voltage collapse which are generally accepted or useful. Existing characterizations of these processes remain primarily in terms of relatively simple paradigms. One consequence of this situation is the continuing discussion on whether voltage collapse is a static or dynamic phenomenon (Tamura *et al.* [26]).

Perhaps the most widely held notion of steady-state stability requires the existence of an equilibrium point, i.e., a solution to the load flow equations, which is stable in the

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sense of Liapunov. Several investigators have considered extentions to the primitive concept of steady-state stability in this spirit, including Venikov *et al.* [23] and Galiana and Lee [24]. It is expected that such a generalized definition could also embrace voltage collapse phenomena. However, voltage sensitivities to various system parameters are typically suggested as "indicators" of voltage collapse, if not as a basis for its definition (e.g., Abe *et al.* [25] and Galiana [35]), but sensitivity considerations do not follow from existing generalized definitions of steady-state stability. In this paper, we propose a definition of static stability of electric power systems which does encompass conventional notions of both steady-state stability and voltage collapse.

Electric power systems operate under the influence of numerous parameters which vary with time and circumstances. As these parameters change, it is generally accepted (see Venikov *et al.* [23]) that the power system can lose stability in one of two ways; either with the abrupt appearance of self-sustained oscillations or with the disappearance of the equilibrium point. The latter is referred to as divergence or as a static bifurcation. In this paper, we study static bifurcations of electric power systems. Our analysis is based on the theory of generic bifurcations which is readily accessible in the recent books by Chow and Hale [17], Vanderbauhede [18], Arnold [19], Guckenheimer and Holmes [20], and Golubitsky and Schaeffer [21]. This theory allows development of a classification of the mechanisms by which an electric power system can be expected to lose static stability.

There is ample evidence relating steady-state stability and voltage collapse with the possible existence of neighboring multiple equilibria (see, for example, Tavora and Smith [22], Venikov *et al.* [23], Galiana and Lee [24], Abe *et al.* [25], and Tamura *et al.* [26]). Thus, static bifurcation theory appears to be an appropriate mathematical framework for investigating these issues. The earliest application of bifurcation theory to power system analysis of which we are aware is the interesting study of a conservative, three-machine study reported by Andronov and Neimark in 1961. Their paper is summarized by Aronovich and Kartvelishvili in the English translation [31]. A bifurcation analysis of the real (active) power flow equation was initiated by Aroposthatis, Sastry, and Varaiya [27]. In this paper, we deal with a much larger class of nonconservative models which includes transfer conductances as well as voltage variations and reactive power effects. The latter is essential to a discussion of voltage collapse. In [33], Sallam and Dineley give an interpretation of the steady-state stability of a single-machine, infinite-bus system in terms of catastrophe theory.

In the following analysis, we consider the standard swing equation model used in power system transient stability analysis with transfer conductances and including *PV* buses (voltage-controlled type), and *PQ* buses (constant power type). In Section II, the model is described in detail. The model is comprised of a coupled system of differential and algebraic equations (sometimes called a semi-implicit or constrained [36] system of differential equations), a point which is central to our analysis. The fact that such semi-implicit systems arise in power system analysis was pointed out by Korsak [37]. We also introduce in Section II the concept of a "regular" equilibrium point and identify an issue of causality in power systems which proves to be an essential aspect of static stability.

Section III consists of a summary of the relevent concepts and results of static bifurcation theory and Section IV summarizes the Liapunov-Schmidt analysis as we have implemented it. In Section V, we define an equilibrium point to be statically stable if it is regular and stable in the sense of Liapunov. It is shown that nonregular equilibria are either static bifurcation points of the load flow equations or are equilibria at which the power system model is not locally causal. We show how existing indicators of steady-state stability and voltage collapse follow from our definition and the most basic results of bifurcation theory. An example is given which illustrates all concepts introduced and several different mechanisms of static instability.

The main conclusion is that both steady-state stability and voltage collapse can be viewed in terms of a common mathematical structure which predicts a much richer set of mechanisms of power system static instability than previously recognized. The painful algebraic calculations required—even in our very modest examples—to carry out the essential Liapunov-Schmidt reduction and subsequent series expansions are facilitated by the use of the MACSYMA symbolic computation program. Our study indicates that static bifurcation theory combined with symbolic computation can be developed into a practical adjunct to load flow analysis for the detailed investigation of situations in which loss of steady-state stability or voltage collapse is a concern.

II. The Power System Model and Local Stability Properties

We consider a system with $n+m$ buses in which buses $i=1,\cdots,n$ are the internal buses of n generators, buses $i=n+1,\cdots,n+m$ are m voltage-controlled (PV) load buses and buses $i=n+m+1,\cdots,n+m+l$ are l PQ load buses. As is usual, we consider this network to be the equivalent reduced network resulting from elimination of the constant admittance loads and internal buses. A complete set of equations describing this system is

$$M_i\delta_i''+D_i\delta_i'+f_i=P_{mi},\quad i=1,\cdots,n \tag{1}$$

$$f_i=P_{li},\quad i=n+1,\cdots,n+m \tag{2}$$

$$f_i=P_{li},\ g_i=Q_{li},\quad i=n+m+1,\cdots,n+m+l \tag{3}$$

where

$$f_i=\sum_j\left[E_iE_jB_{ij}\sin(\delta_i-\delta_i)+E_iE_jG_{ij}\cos(\delta_i-\delta_i)\right],\quad i=1,\cdots,n+m+l \tag{4}$$

$$g_i=\sum_j\left[E_iE_jG_{ij}\sin(\delta_i-\delta_j)-E_iE_jB_{ij}\cos(\delta_i-\delta_j)\right],\quad i=n+m+1,\cdots,n+m+l \tag{5}$$

and M_i denotes the rotor inertia, D_i the damping constant, E_i the bus voltage, δ_i the bus voltage angle, $B_{ij}(G_{ij})$ the transfer susceptance (conductance), P_{mi} the turbine mechanical power injection, P_{1i} and Q_{1i} the load real and reactive power components.

Equations (1) and (2) can be placed in a convenient standard form by defining the vectors

$$\delta = [\delta_1, \cdots, \delta_n]' \tag{6}$$

$$\delta_l = [\delta_{n+1}, \cdots, \delta_{n+m+l}]' \tag{7}$$

$$E = [E_{n+m+1}, \cdots, E_{n+m+l}]' \tag{8}$$

$$y = [\delta_l', E']' \tag{9}$$

and the functions

$$f_1(\delta, y, \lambda) = [f_1 - P_{m1}, \cdots, f_n - P_{mn}]' \tag{10}$$

$$\begin{aligned} f_2(\delta, y, \lambda) &= \left[f_{n+1} - P_{l(n+1)}, \cdots, f_{n+m+1} - P_{l(n+m+l)}, g_{n+m+l} \right. \\ &\quad \left. - Q_{l(n+m+1)}, \cdots, g_{n+m+1} - Q_{l(n+m+l)} \right]' \end{aligned} \tag{11}$$

$$f(\delta, y, \lambda) = [f_1', f_2']' \tag{12}$$

so that (1) and (2) can be written

$$M\delta'' + D\delta' + f_1(\delta, y, \lambda) = 0 \tag{13a}$$

$$f_2(\delta, y, \lambda) = 0. \tag{13b}$$

An *equilibrium point of (13)* is a point $(\delta^*, y^*, \lambda^*)$ that satisfies the relation $f(\delta^*, y^*, \lambda^*) = 0$. We will call the system (locally) *causal at* $(\delta^*, y^*, \lambda^*)$ if (13b) admits a solution $y(\delta, \lambda^*)$ in a neighborhood of (δ^*, y^*) with $y(\delta^*, \lambda^*) = y^*$, and *strongly causal at* $(\delta^*, y^*, \lambda^*)$ if it has a solution $y(\delta, \lambda)$ in a neighborhood of $(\delta^*, y^*, \lambda^*)$ with $y(\delta^*, \lambda^*) = y^*$. It is *strictly causal* if it is strongly causal and the solution $y(\delta, \lambda)$ is unique. These definitions are motivated by the fact that if (13b) does not admit a local solution for y, then the state trajectories of (13) must be interpreted as distributions since they contain singularity functions. Moreover, in this case, the machine angle trajectories $\delta(t)$ are not uniquely determined by the initial "state" $\delta(t_0), \delta'(t_0)$. We will be primarily concerned with the properties of (13) in the neighborhood of equilibria at which (13) is causal. If (13) is not causal, then the validity of the model as a characterization of the power system is questionable. It is likely that parasitic effects, neglected in (13), now become central to the local power system behavior.

An equilibrium point $(\delta^*, y^*, \lambda^*)$ is *regular* if there are unique (modulo the translational symmetry) functions $\delta_e(\lambda)$ and $y_e(\lambda)$ satisfying $f(\delta_e, y_e, \lambda) = 0$ in a neighborhood of $(\delta^*, y^*, \lambda^*)$, such that $\delta^* = \delta_e(\lambda^*)$, $y^* = y_e(\lambda^*)$, and (13) is strongly causal at $(\delta^*, y^*, \lambda^*)$. The following theorem will be useful.

Theorem 1: If $(\delta^*, y^*, \lambda^*)$ is a regular equilibrium point then $D_y f_2(\delta^*, y^*, \lambda^*)$ has an inverse.

Proof: Since the system is strongly causal at the equilibrium point, $f_2 = 0$ has a solution $y(\delta, \lambda)$ in a neighborhood of $(\delta^*, y^*, \lambda^*)$ with $y(\delta^*, \lambda^*) = y^*$. To demonstrate that the inverse exists, we need only show that the solution is unique. Note that a solution exists iff

$$\operatorname{rank}\{D_\delta f_2 \quad D_y f_2 \quad D_\lambda f_2\}^* = \operatorname{rank}\{D_y f_2\}^*.$$

Thus, the maximum $\operatorname{rank}\{D_\delta f_2 D_y f_2 D_\lambda f_2\}^* = m$ occurs when $\{D_y f_2\}^*$ has an inverse and the solution is unique. Now, at a regular equilibrium point, we must have

$$\operatorname{rank}\begin{Bmatrix} D_\delta f_1 & D_y f_1 & D_\lambda f_1 \\ D_\delta f_2 & D_y f_2 & D_\lambda f_2 \end{Bmatrix}^* = n + m - 1$$

which is its maximum rank and this can only occur if $\operatorname{rank}\{D_\delta f_2 \quad D_y f_2 \quad D_\lambda f_2\}^* = m$.

Corollary 1: The system (13) is strictly causal at a regular equilibrium point.

If (13) is strictly causal at the equilibrium point $(\delta^*, y^*, \lambda^*)$, the linearized dynamics of (13) reduce to the differential equation

$$Mx'' + Dx' + Kx = 0 \tag{14}$$

where

$$K = \left\{ D_\delta f_1 - D_y f_1 (D_y f_2)^{-1} D_\delta f_2 \right\}^* \tag{15}$$

and $x = \delta - \delta^*$. The results of Kwatny, Bahar, and Pasrija [29] provide a complete characterization of the stability of regular equilibria. In general, $M' = M > 0$, $D' = D \geqslant 0$. However, K is not typically symmetric. The essential stability properties of (14) are provided in [29]. We summarize some of the results of [29] in the following theorem.

Theorem 2: The system (14) is stable for all $D \geqslant 0$ only if there exists a real, symmetric, positive-definite symmetrizing matrix S such that $SK = Q$, where $Q = Q'$ has precisely one zero eigenvalue corresponding to the translational symmetry and its remaining $n-1$ eigenvalues are positive.

Remark: This is a necessary but not sufficient condition for stability. If D is restricted so that $M^{-1}D$ and $M^{-1}K$ commute (for example, uniform damping) then it is sufficient as well.

If (13) is not strictly causal, then the linearized dynamics are of the singular implicit type (see Verghese *et al.* [38] and Bernhard [39]) and are not reducible to the form of (14). Local behavior can be analyzed using the methods of [38] and [39]; however, we will not consider this point here.

Although our discussion will center on (13), it will sometimes be convenient to interpret our results and compare them with those of others in terms of the physical variables. For this purpose, define the functions

$$f_g(\delta, \delta_1, E, \lambda) = [f_1, \cdots, f_n]' - P_m \tag{16}$$

$$f_l(\delta, \delta_1, E, \lambda) = [f_{n+1}, \cdots, f_{n+m+l}]' - P_l \tag{17}$$

$$g_l(\delta, \delta, E, \lambda) = [g_{n+m+1}, \cdots, g_{n+m+l}]' - Q_l. \tag{18}$$

Equations (13) can now be expressed

$$M\delta'' + D\delta' + f_g(\delta, \delta_l, E, \lambda) = 0 \tag{19a}$$

$$f_l(\delta, \delta_l, E, \lambda) = 0 \tag{19b}$$

$$g_l(\delta, \delta_l, E, \lambda) = 0. \tag{19c}$$

The standard load flow equations are obtained from (19) by setting δ'', δ' equal to zero. Since these equilibrium equations have a translational symmetry in the angle variables, it is commonplace to measure all angles relative to an arbitrarily selected *swing* bus. Thus, we will specify bus number 1 as the swing bus (any other generator bus will do), and define the generator and load relative angle vectors

$$\theta = (\delta_1, \delta_2 - \delta_1, \cdots, \delta_n - \delta_1)' \tag{20a}$$

$$\phi = (\delta_{n+1} - \delta_1, \cdots, \delta_{n+m+1} - \delta_l)' \tag{20b}$$

and the functions

$$F_1(\theta, \phi, E, \lambda) = (f_1 - P_{m1}, \cdots, f_n - P_{mn})'|_{\delta, \delta_l \to \theta, \phi} F_2(\theta, \phi, E, \lambda) \tag{21a}$$

$$= (f_{n+1} - P_{1(n+1)}, \cdots, f_{n+m+l} - P_{1(n+m+l)})'|_{\delta, \delta_l \to \theta, \phi} \tag{21b}$$

$$F_3(\theta, \phi, E, \lambda) = (g_l - Q_l)|_{\delta, \delta_1 \to \theta, \phi} \tag{21c}$$

$$F = [F_1' \;\; F_2' \;\; F_3']' \tag{22}$$

so that equilibria satisfy the load flow equation

$$F(\theta, \phi, E, \lambda) = 0. \tag{23a}$$

Translational symmetry implies that $F(\theta, \phi, E, \lambda)$ is independent of θ_1. Let θ_r denote the reduced vector $(\theta_2, \cdots, \theta_n)$. By dropping the first equation of (23a), the load flow equations can be written in reduced form

$$F_r(\theta_r, \phi, E, \lambda) = 0. \tag{23b}$$

Remark: It is tacitly assumed that any solution of (23b) automatically satisfies the first equation of (23a) and is therefore a valid solution of (23a). When (23b) has multiple solutions, it is reasonable to ask whether all of them are legitimate solutions to (23a). We remark that it is always possible (ignoring physical limitations) to adjust the mechanical power input to generator number one so that the omitted equation is satisfied. In other words, the first equation of (23a), F^1, has a distinguished parameter, namely P_{m1}, which does not appear in any other equation, and it is possible to determine the function $P_{m1}(\theta_r, \phi, E, \lambda)$ so that $F^1(\theta_r, \phi, E, \lambda, P_{m1}(\theta_r, \phi, E, \lambda)) = 0$ when (23b) is satisfied. This, in turn implies that dF^1 is a linear combination of the rows of dF_r.

It is convenient to use (23b) to study the dependence of the equilibria on the parameters λ; however, (14) remains the basis for determining the stability of an equilibrium point. A straightforward calculation provides the following result which connects the stability characteristics of (14) to the function $F_r(\theta_r, \phi, E, \lambda)$. First we recall from [29] the following theorem.

Theorem 3: If a real matrix A is symmetrizable by a matrix S, as in Theorem 2, so that $SA = Q$, then A has independent eigenvectors, real characteristic roots, and these roots have the same sign as Q.

Now, we are in a position to state the following theorem.

Theorem 4: At a strictly causal equilibrium point, a symmetrizable matrix K of (15) has one zero eigenvalue corresponding to the translational symmetry. It has additional zero eigenvalues if and only if the matrix K_r is singular, where

$$K_r = \left\{ D_{\theta_r} F_{r1} - [D_\phi F_{ri} D_E F_{r1}] \cdot \begin{bmatrix} D_\phi F_{r1} & D_E F_{r2} \\ D_\phi F_{r3} & D_E F_{r3} \end{bmatrix}^{-1} \begin{bmatrix} D_{\theta_r} F_{r2} \\ D_{\theta_r} F_{r_3} \end{bmatrix} \right\}^* . \tag{24}$$

Proof: At a strictly causal equilibrium point, there exists a solution $y = y(\delta, \lambda)$ of $f_2(\delta, y, \lambda) = 0$. Define the function $h(\delta, \lambda) = f_1(\delta, y(\delta, \lambda), \lambda)$. It is easily verified that $h(\delta, \lambda)$ inherits the translational symmetry in δ of $f(\delta, y, \lambda)$. Define the function $H(\theta, \lambda) = h(\Phi\theta, \lambda)$, where $\delta = \Phi\theta$ is the transformation defined in (20a). Note that $D_\theta H^* = D_\delta h^* \Phi$. Let S be the symmetrizing matrix for $K = D_\delta h^*$. We can write

$$(\Phi' S \Phi)\Phi^{-1} D_\theta H^* = \Phi' S K \Phi. \tag{25}$$

Thus, it is clear that $\Phi^{-1} D_\theta H^*$ is symmetrizable and, hence, has real eigenvalues. Moreover, according to Theorem 3, the signs of the roots of $\Phi^{-1} D_\theta H^*$ correspond to those of $\Phi' S K \Phi$. But the roots of $\Phi' S K \Phi$ are identical to those of SK, and Theorem 3 again implies that these roots have signs corresponding to the roots of K. Thus, we have shown that the eigenvalues of $\Phi^{-1} D_\theta H^*$ have signs corresponding to those of K.

Now, let K have an eigenvector corresponding to a zero eigenvalue other than the translational symmetry; then there is a corresponding eigenvector of $\Phi^{-1} D_\theta H^*$ and, hence, of $D_\theta H^*$. It is convenient to denote this eigenvector by $e = [e_1 e_r']'$. Translational symmetry corresponds to $e_1 \neq 0$ and $e_r = 0$. Since $\Phi^{-1} D_\theta H^*$ is symmetrizable, e is independent of the translational symmetry eigenvector so that e_r must be nontrivial. Direct computation shows that $D_\theta H^* e = 0$ implies $K_r e_r = 0$ and necessity is proved.

For sufficiency, assume that there exists a nontrivial e_r so that $K_r e_r = 0$. Not that $D_\theta H^* = [0 | D_{\theta_r} H^*]$ and $D_{\theta_r} H^* = [b' | K_r']'$. Furthermore, in view of the remark following (23b), the row b is a linear combination of the rows of K_r. Thus, $D_{\theta_r} H^* \; e_r = 0$ and $D_\theta H^*$ has a zero eigenvalue corresponding to nontrivial e_r. It follows from the arguments above that K has a zero eigenvalue which does not correspond to translational symmetry.

Remark: The significance of this theorem is that it provides a necessary and sufficient condition for incipient divergence instability.

III. Static Bifurcation

The power system equilibrium equations typically depend on a very large number of parameters. Moreover, the number of parameters differs from system to system and from time to time. The essential problem of the analysis of power system static stability is to recognize impending change in system behavior as these parameters vary and to identify the controlling parameters. Bifurcation theory provides the tools to deal with such issues. A summary of the necessary ideas follows.

Let X, Z be Banach spaces, $\Omega \subset X$, an open set, and let $C^r(\Omega, Z)$ denote the Banach space of r-times continuously differentiable functions $f\colon \Omega \to Z$ satisfying

$$|f|_r = \sup\{|f(x)| + |Df(x)| + \cdots + |D^r f(x)| : x \in \Omega\} < \infty.$$

Let $r \geqslant 1$ and let $\tau \subset C^r(\Omega, Z)$ be a given family in $C^r(\Omega, Z)$. Suppose there exists $x_0 \in \Omega$ and $f_0 \in \tau$ such that

$$f_0(x_0) = 0.$$

Then, (f_0, x_0) is a *bifurcation point* with respect to τ if in each neighborhood U of (f_0, x_0) in $C^r(\Omega, Z) \times X$ there exists (f, x_1) and (f, x_2) with $f \in \tau$, $x_1 \neq x_2$ and $f(x_1) = 0$, $f(x_2) = 0$.

Thus, (f_0, x_0) is a bifurcation point relative to a family τ if an arbitrarily small perturbation of $f_0 \in \tau$, also belonging to the family τ, has multiple zeros near x_0. The central problem of bifurcation theory is to characterize the zeros of $f \in \tau$ in a given neighborhood U of (f_0, x_0) in $C^r(\Omega, Z) \times X$. It is useful to distinguish two problems (Hale [1978]).

Generic bifurcation problem—Given a neighborhood U of (f_0, x_0) in $C^r(\Omega, Z) \times X$, characterize the number of zeros of (f, x) in U.

Restricted bifurcation problem—Given a neighborhood U of (f_0, x_0) in $C^r(\Omega, Z) \times X$ and a family $\tau \subset C^r(\Omega, Z)$, characterize the number of zeros of $f \in \tau$, and (f, x) in U.

The importance of this distinction rests with the fact that these problems represent two points of view which lead naturally to substantially different methods of attack. In order to make this point clear, we briefly review the approach to each problem. We note that, in a particular application, one or the other of these viewpoints may be preferred.

Before proceeding, it is convenient to recast the above formulation of the bifurcation problem into an equivalent but perhaps more conventional formalism. Let $\Lambda = C^r(\omega, Z)$ and define

$$F(x, f) = f(x), \qquad \text{for all } x \in \Omega \text{ and } f \in C^r(\Omega, Z).$$

Clearly, there is a $\Gamma \subset \Lambda$ such that the family τ can be defined by the relation

$$\tau = \{F(x, \lambda) : x \in \Omega, \lambda \in \Gamma\}.$$

By associating λ_0 with f_0, the point (x_0, λ_0) is a bifurcation point and the bifurcation problem is to characterize the solution set of

$$F(x, \lambda) = 0$$

in a given neighborhood of (x_0, λ_0) in $\Omega \times \Lambda$ (the generic problem) or in $\Omega \times \Gamma$ (the restricted problem). It is usual to refer to λ as the parameter and to λ_0 as a *bifurcation value* of the parameter. In our applications, we will always take Ω and Γ to be open subsets of R^n and R^k, respectively, and Z will be R^n.

Note that if the derivative $A = D_x F(x_0, \lambda_0)$ has a bounded inverse, then the implicit function theorem guarantees a unique zero in a neighborhood of x_0 for each λ in a neighborhood of λ_0. It follows that (x_0, λ_0) cannot be a bifurcation point. Thus, we assume that A does not have a bounded inverse. Let $\mathrm{Ker}(A)$ denote the null space of A and $\mathrm{Im}(A)$ its range and further assume that $\dim[\mathrm{Ker}(A)] = p = \mathrm{codim}[\mathrm{Im}(A)]$. Then, using the method of Liapunov-Schmidt (to be described in some detail below), the study of the zeros of $F(x, \lambda)$ can be reduced to the study of the zeros of the so-called *bifurcation equation*, which represents p equations in p unknowns. Indeed, the bifurcation equation can be written

$$\phi(u, \lambda) = 0 \quad \phi\colon R^p \times V \to R^p, \qquad V \subset \Lambda.$$

Now, for suitably smooth functions, the theory of universal unfoldings deals with the question of determining, by appropriate transformations in the space of mappings and the u-space, a function $\phi^*(u^*, \gamma(\lambda))$ such that the zeros of ϕ^* coincide with those of ϕ, the parameter γ is of finite dimension with some minimum dimension q, and ϕ^* is a polynomial of some degree in u^*. Thus, the characterization of the zeros of the function $F(x, \lambda)$ is reduced to the study of the much simpler problem

$$\phi^*(u^*, \gamma) = 0 \quad \phi^*\colon R^p \times R^q \to R^p.$$

The function ϕ^* is the universal unfolding of the function ϕ at the bifurcation point end and it allows a complete characterization of the zeros of $F(x, \lambda)$ for λ in a neighborhood of λ_0. The dimension of the γ-space is a measure of the *degeneracy* of the bifurcation. It follows from analysis of the polynomial function ϕ^* that there exists a neighborhood W of $\gamma_0 = \gamma(\lambda_0)$ in γ-space, which is divided into finitely many open regions by surfaces of co-dimension 1 such that throughout each region $\phi^*(u^*, \gamma)$ has the same number of zeros. The surfaces across which the number of zeros change are called *bifurcation surfaces*. These bifurcation surfaces can intersect, thereby defining (bifurcation) surfaces of higher codimension. The point γ_0 lies at an intersection of codimension q in γ-space.

The concept of universal unfolding seems to have originated with Thom and has been agressively developed by many other investigators. In the case that ϕ is derivable from a potential function (reducing the problem to that of catastrophe theory), the theory is extensive and the universal unfoldings have been classified. The case $p = 1$ always fits into this category. Some results are also available for the general bifurcation problem (that is, ϕ is not the gradient of a potential function) when $p = 2$. As pointed out by Hale [30], even when p is small the degeneracy of the bifurcation, when viewed as a generic bifurcation, can be inconveniently large. Thus, when $p > 1$, it is probably advantageous to assume the point of view of the restricted bifurcation problem.

We introduce the restricted bifurcation problem with a qualitative description of bifurcation in the original parameter space Λ (see Arnold [19] for a more complete discussion). There exists a neighborhood V of λ_0 in Λ

which is divided into open regions by codimension 1 surfaces. Throughout each region, $F(x,\lambda)$ has the same number of zeros. Since λ_0 corresponds to a bifurcation point, it lies on one of the codimension 1 manifolds which form the boundaries of the open sets containing the generic points. These manifolds of singular points may intersect forming manifolds of codimension 2 or greater in V. If λ_0 lies on an intersection of codimension q, then the bifurcation is said to be a *degeneracy of codimension q*.

Suppose now that we consider a k-parameter family of functions τ. That is, Γ is a k-dimensional manifold in Λ. The family τ contains a bifurcation with degeneracy of codimension q if Γ intersects a manifold of singular points of codimension q. If the k-dimensional manifold Γ intersects a manifold of codimension $q>k$, then the intersection can be removed by an arbitrarily small deformation of the family τ. Thus, only bifurcations with degeneracy of k or less are generic in k-parameter families. We call a k-parameter family a *generic family* if it contains only bifurcations with degeneracy k or less —or, more precisely, all bifurcations correspond to transversal intersections of Γ with bifurcation surfaces of codimension k or less. The following theorem is a useful characterization of generic families (Arnold [19]).

Theorem 5: For a one parameter generic family, the set of solutions of $F(x,\lambda)=0$ is a smooth manifold in $\Omega\times\Gamma$.

Thus, at all points on the solution manifold of a generic family, we have

$$\operatorname{rank}[D_xF|D_\lambda F]=n.$$

There are other generic properties of restricted bifurcation problems (for example, see Arnold [19], Golubitsky and Schaeffer [21]).

Theorem 6: For one parameter generic families, all bifurcation values are isolated and each bifurcation value corresponds to exactly one bifurcation point with nonzero second differential.

IV. The Liapunov-Schmidt Reduction

The essentials of the method of Liapunov-Schmidt for the analysis of bifurcation problems will be briefly reviewed. Further details and examples may be found in [17] and [21]. We consider a smooth map $F\colon \Omega\times V\to Z$, and for convenience we assume $(0,0)\in\Omega\times V$ and is a bifurcation point. Define $A=D_xF(0,0)$ so that

$$F(x,\lambda)=Ax+N(x,\lambda) \tag{26}$$

where

$$N(0,0)=0 \quad D_xN(0,0)=0. \tag{27}$$

Thus, we wish to study the solution set of the equation

$$Ax+N(x,\lambda)=0 \tag{28}$$

in a neighborhood of (0,0). Let $P\colon X\to X$, and $Q\colon Z\to Z$ denote projection operators and let X_P and Z_Q denote $\operatorname{Im}(P)$ and $\operatorname{Im}(Q)$, respectively. The following theorem is the key to the reduction method.

Theorem 7: If

$$\operatorname{Ker}(A)=X_P \text{ and } \operatorname{Im}(A)=Z_Q$$

then there exists a bounded linear operator $K\colon Z_Q\to X_{I-P}$, called the right inverse of A, such that $AK=I$ on Z_Q and $KA=I-P$ on X and (28) is equivalent to

$$v-KQN(u+v,\lambda)=0 \tag{29a}$$

$$(i-Q)N(u+v,\lambda)=0 \tag{29b}$$

where $x=(u+v,\ u=P\in X_P$ and $v=(I-P)\in X_{I-P}$.

Applying the Implicit Function Theorem to (29a), there is a (smooth) unique function $v^*(u,\lambda)$ on a neighborhood of (0,0). Thus, on a neighborhood of (0,0), equation (28) is equivalent to

$$(I-Q)N(u+v^*(u,\lambda),\lambda)=0. \tag{30}$$

Equation (30) is referred to as the *(reduced) bifurcation equation*. Note that the number of independent equations represented by (30) is $\dim[u]=\dim[\operatorname{Ker}(A)]$.

A complete analysis can be given for the case

$$\dim[\operatorname{Ker}(A)]=1=\operatorname{codim}[\operatorname{Im}(A)]. \tag{31}$$

Our objective is to characterize the number of zeros in a neighborhood of the bifurcation point. The essential results are summarized below. Let u_0, w_0 be nontrivial vectors with $u_0\in\operatorname{Ker}(A)$ and $w_0\in(I-Q)Z$ (recall that $\dim[(I-Q)Z]=\operatorname{codim}[(A)]$). Then, we can write $u=au_0$ and obtain the function $\nu^*(a,\lambda)=v^*(au_0,\lambda)$. Now, the pair (x,λ) satisfies (28) in a neighborhood of (0,0) if $x=au_0+v, v=\nu^*(a,\lambda)$ and the pair (a,λ) satisfies the bifurcation equation

$$\phi(a,\lambda)=0 \tag{32}$$

where the function $\phi(a,\lambda)$ is defined by

$$\phi(a,\lambda)w_0=(I-Q)F(au_0+\nu^*(a,\lambda),\lambda). \tag{33}$$

It is easily verified that

$$\phi(0,0)=0 \text{ and } D_a\phi(0,0)=0. \tag{34}$$

Suppose, further, that the first nonvanishing derivative is the kth. That is

$$D_a^i\phi(0,0)=0, \qquad i=0,\cdots,k-1 \tag{35a}$$

$$D_a^k\phi(0,0)\neq 0. \tag{35b}$$

As is well known from singularity theory, under certain genericity conditions to be discussed below, (32) is locally equivalent to its universal unfolding

$$\phi^*(z,\gamma)=\gamma_0+\gamma_1z+,\cdots,+\gamma_{k-2}z^{k-2}+z^k \tag{36}$$

where $z=z(a,\lambda)$ and $\gamma_i=\gamma_i(\lambda)$ are smooth, near identity transformations. In actual computation, the transformations can be readily determined using the Taylor expansion of $\phi(a,\lambda)$.

Equation (36) can be used to characterize the number of zeros near the bifurcation point. The basic idea is to classify the number of zeros of ϕ^* in the γ-space, that is, in terms of the $k-1$-dimensional parameter vector $\gamma=(\gamma_0,\gamma_1,\cdots,\gamma_{k-2})^t$. The number of solutions change when the parameter value crosses a singular (bifurcation) surface of multiple solutions. Multiple solutions occur when the following hold:

$$\phi^*(z,\gamma)=0 \tag{37}$$

$$D_z\phi^*(z,\gamma)=0. \tag{38}$$

In principle, the codimension 1 bifurcation surfaces can be found by eliminating z between (37) and (38), thereby obtaining an algebraic relation involving only the parameters. As will be seen, it is easier to derive a parametric representation of the bifurcation surfaces.

Specific results for $k=2$, 3, 4 will be summarized. Further details may be found in Chow and Hale [17].

$$k=2 \qquad \phi^*(z,\gamma)=\gamma_0+z^2 \tag{39}$$

from which it is concluded that $\gamma_0=0$ defines the bifurcation surface, a point in the 1-dimensional γ-space, and

(i) $\gamma_0>0$ implies no solution of (37)
(ii) $\gamma_0=0$ implies one solution of (37)
(iii) $\gamma_0<0$ implies two solutions of (37).

$$k=3 \qquad \phi^*(z,\gamma)=\gamma_0+\gamma_1 z+z^3 \tag{40}$$

from which

$$D_z\phi^*(z,\gamma)=\gamma_1+3z^2$$

and (37) and (38) yield the parametric characterization of the bifurcation surfaces

$$\gamma_0=-3z^2$$
$$\gamma_1=2z^3.$$

These relations constitute a parametric representation of a cusp in the γ-plane.

$$k=4 \qquad \phi^*(z,\gamma)=\gamma_0+\gamma_1 z+\gamma_2 z^2+z^4. \tag{41}$$

The bifurcation surfaces are defined by the simultaneous solution of the equations

$$\gamma_0+\gamma_1 z+\gamma_2 z^2+z^4=0$$
$$\gamma_1+2\gamma_2 z+4z^3=0$$

which define surfaces in three-dimensional γ-space called the swallow tail.

A brief description of the required conditions follows. Let W be a neighborhood of the origin of R^{k-1}. In the original parameter space Λ, the family $\phi^*(z,\gamma)$ is represented by a smooth $(k-1)$-dimensional manifold Γ, containing the origin of Λ, and such that

$$W=\{\gamma\in R^{k-1}|\gamma=\gamma(\lambda),\lambda\in\Gamma\}. \tag{42}$$

It is easy to see that such a manifold exists in a neighborhood of the origin iff

$$\operatorname{rank}[D_\lambda\gamma(0)]=k-1. \tag{43}$$

The manifold B in Λ defined by

$$B=\{\lambda\in\Lambda|\gamma(\lambda)=0\} \tag{44}$$

is the bifurcation surface corresponding to the set of bifurcation parameters of maximum degeneracy $(k-1)$. It is the intersection of bifurcation surfaces of lower degeneracy. Clearly, B is of codimension $(k-1)$ and Γ and B intersect at the origin. Moreover, it is easily shown that the intersection is transversal. Relation (43) is called a generic condition and is necessary for the bifurcation to be generic. Note that (43) requires that $\dim(\lambda)\geqslant k-1$.

V. Static Bifurcation in Power Systems

We are now in a position to examine the following concept of static stability of electric power networks. An equilibrium point is *statically stable* if it is regular and stable in the sense of Liapunov.

Note that, by definition, a bifurcation point of the load flow equations cannot be a regular equilibrium point. Therefore, a bifurcation point is not statically stable. By Corollary 1, a regular equilibrium point is strictly causal. Therefore, an equilibrium point which is not strictly causal is not regular and, hence, it is not statically stable. We will show, by an example, that bifurcation points may or may not be strictly causal and that equilibria which are not strictly causal need not be bifurcation points. Changes in the causal properties of equilibria under parameter variations may be studied by analyzing bifurcations of the subset of the load flow equations (21b and c). We will not pursue such an analysis here.

An equilibrium point of the electric power system $(\theta_r^*,\phi^*,E^*,\lambda^*)$ satisfies the load flow equation (23b). It is a bifurcation point only if the Jacobian

$$J_r=\begin{bmatrix}D_{\theta_r}F_r & D_\phi F_r & D_E F_r\end{bmatrix} \tag{45}$$

is singular at $(\theta_r^*,\phi^*,E^*,\lambda^*)$. Venikov *et al.* [23] recognized the significance of a degeneracy in J_r with respect to the steady-state stability of a power system. They observed that, under certain conditions, a change in the sign of $\deg\{J_r\}$ during a continuous variation of system parameters coincides with the movement of a real characteristic root of the linearized swing equations across the imaginary axis into the right half of the complex plane. Thus, they recommended tracking $\det\{J_r\}$ during load flow calculations and proposed a modification of Newton's method which allows precise determination of the parameter value where such a sign change occurs. Tamura *et al.* [24] discuss some computational experience using this method.

In the terminology of this paper, the basis for their argument is easily established. Let

$$A=\begin{bmatrix}D_\phi F_{r2} & D_E F_{r2}\\ D_\phi F_{r3} & D_E F_{r3}\end{bmatrix}.$$

Suppose that $\det\{J_r\}=0$ at λ^* and $\det\{A\}\neq 0$ for all λ in a neighborhood of λ^* (i.e., the equilibrium point is strictly causal). Using Schur's formula, we have

$$\det\{J_r\}=\det\{K_r\}\det\{A\}$$

so that the change in sign of $\det\{J_r\}$ at λ^* coincides with a change in sign of $\det\{K_r\}$, which typically corresponds to a single real root of K_r crossing the imaginary axis.

A further elaboration of the significance of a degeneracy in J_r has been given by Abe *et al.* [25]. To summarize their arguments, first note that (23b) yields the following fundamental relation:

$$D_{\theta_r} F_r d\theta + D_\phi F_r d\phi + D_E F_r dE + D_\lambda F_r d\lambda = 0. \quad (46)$$

Now, partition J_r and $D_\lambda F_r$ according to the following definitions:

$$A_1 = \begin{bmatrix} D_{\theta_r} F_{r1} & D_\phi F_{r1} \\ D_{\theta_r} F_{r2} & D_\phi F_{r2} \end{bmatrix} \qquad A_2 = \begin{bmatrix} D_E F_{r1} \\ D_E F_{r2} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} D_{\theta_r} F_{r3} & D_\phi F_{r3} \end{bmatrix} \qquad A_4 = [D_E F_{r3}]$$

$$B_1 = D_\lambda \begin{bmatrix} F_{r1}^t & F_{r2}^t \end{bmatrix}^t \qquad B_2 = D_\lambda F_{r3}.$$

If $\det\{A_4\} \neq 0$, then applicator of Schur's formula leads to

$$\det\{J_r\} = \det\{A_4\} \det\{A_1 - A_2 A_4^{-1} A_3\}. \quad (47)$$

Also, in this case, we obtain from (46) the relation

$$[A_1 - A_2 A_4^{-1} A_3][d\theta_r^t \quad d\phi^t]^t = -[B_1 - A_2 A_4^{-1} B_2]\, d\lambda. \quad (48)$$

From (47), it is clear that $(\theta_r^*, \phi^*, E^*, \lambda^*)$ is a bifurcation point only if $\det\{A_1 - A_2 A_4^{-1} A_3\}^* = 0$. Equation (48) shows that this condition implies that the angle variables are "infinitely sensitive" to small changes in the parameters. This property is generally associated with the phenomenon referred to as loss of steady-state stability.

On the other hand, suppose $\det\{A_1\} \neq 0$, so that application of Schur's formula provides

$$\det\{J_r\} = \det\{A_1\} \det\{A_4 - A_3 A_1^{-1} A_2\} \quad (49)$$

and (46) yields

$$[A_4 - A_3 A_1^{-1} A_2]\, dE = -[B_2 - A_3 A_1^{-1} B_1]\, d\lambda. \quad (50)$$

Thus, a bifurcation occurs only when $\det\{A_4 - A_3 A_1^{-1} A_2\}^* = 0$, and, in this case, the bifurcation is associated with infinite sensitivity of the load voltage magnitudes with respect to parameter perturbations. This property is the essential feature of so-called voltage collapse. In fact, it is sometimes used as the definition of voltage collapse.

The Jacobian J_r can of course be singular without either A_1 or A_4 being nonsingular or with both A_1 and A_4 nonsingular. Thus, not all bifurcations can be given one of the two more or less conventional interpretations described above. However, Abe *et al.* [25] point out that it is common practice to operate power systems in such a way that the phase angle difference across each transmission lines is less than $\pi/2$ and in this case $\det\{A_1\} \neq 0$.

Example: The following example, adapted from Johnson [28], illustrates several of the concepts developed above in a simple physical context. Fig. 1 shows the three-bus network to be considered.

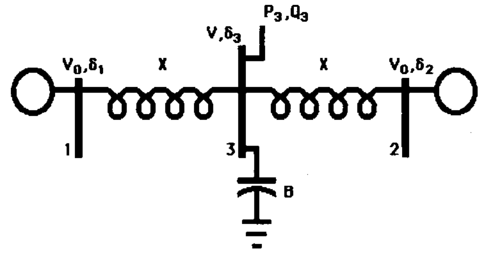

Fig. 1. Network schematic for example.

The system equations are

$$M_1 \delta_1'' + (V_0 V/X) \sin(\delta_1 - \delta_3) = P_1 \quad (51a)$$

$$M_2 \delta_2'' + (V_0 V/X) \sin(\delta_2 - \delta_3) = P_2 \quad (51b)$$

$$(V_0 V/X) \sin(\delta_3 - \delta_1) + (V_0 V/X) \cdot \sin(\delta_3 - \delta_2) = P_3 \quad (51c)$$

$$-(V_0 V/X) \cos(\delta_3 - \delta_1) - (V_0 V/X) \cos(\delta_3 - \delta_2) + (2/X - B) V^2 = Q_3. \quad (51d)$$

The translational symmetry which exists for all values of the parameters implies that solutions exist only if $P_1 + P_2 + P_3 = 0$. We assume that this is the case. For convenience, we fix some of the parameters: $V_0 = 1$, $X = 1$, $M_1 = 1$, $M_2 = 1$. Consistent with our earlier notation, let $\theta = \delta_2 - \delta_1$, $\phi = \delta_3 - \delta_1$, and define $\Delta P = P_2 - P_1$. The reduced equations are

$$\theta'' = -V\sin(\theta - \phi) - V\sin\phi + \Delta P \quad (52a)$$

$$0 = V(\sin\phi + \sin(\phi - \theta)) - P_3 \quad (52b)$$

$$0 = -V(\cos\phi + \cos(\phi - \theta)) + (2 - B)V^2 - Q_3. \quad (52c)$$

Three cases, illustrating three static instability situations, will be discussed.

Case 1: Loss of steady-state stability

parameter values: $\Delta P = \sqrt{2}$, $P_3 = 0$, $B = 2 - \sqrt{2}$, $Q_3 = 0$

equilibrium point: $\theta^* = \pi/2$, $\phi^* = \pi/4$, $V^* = 1$.

Case 2: Voltage collapse

parameter values: $\Delta P = 0$, $P_3 = -1$, $B = 0$, $Q_3 = 0$

equilibrium point: $\theta^* = 0$, $\phi^* = -\pi/4$, $V^* = 1/\sqrt{2}$.

Case 3: Loss of causality

parameter values: $\Delta P = 1$, $P_3 = 1$, $B = 1$, $Q_3 = 0$

equilibrium point: $\theta^* = \pi/2$, $\phi^* = \pi/2$, $V^* = 1$.

Cases 1 and 2 are simple (fold) bifurcations with one-dimensional null space and with $k = 2$ (see (39)). The functions $\gamma_0(\lambda)$ and the null space spanning vectors u_0 are given by

Case 1:

$$\gamma_0 = -\sqrt{2}(B - 2 + \sqrt{2}) + \sqrt{2}(\Delta P - \sqrt{2}) + \sqrt{2}\, Q_3 - (B - 2 + \sqrt{2})^2/8$$

$$u_0 = (1, 1/2, -1/2)^t$$

Case 2:

$$\gamma_0 = -B/4 - \Delta P/2 + (P_3+1)/2 - Q_3/2 - B^2/16$$

$$u_0 = (0, -\sqrt{2}, 1)'.$$

We call Case 1 loss of steady-state stability because it clearly represents a case of maximum real power transfer from bus 2 to bus 1 with a phase-angle difference of $\pi/2$. Straightforward calculation will verify that this equilibrium point is a bifurcation point and that it is strictly causal. Moreover, both A_1 and A_4 are nonsingular.

Case 2 is essentially the situation used by Glavitsch [40] to illustrate voltage collapse. It is a bifurcation point and it is not strictly causal. We will comment further on this case below.

Case 3 illustrates loss of causality. It does not represent a bifurcation of the complete set of load flow equations. Indeed, $\det\{J_r\} \neq 0$. It does represent a bifurcation of (52b and c) with θ treated as a parameter. Analyzing the bifurcation from this point of view, it is again found to be a fold bifurcation with

$$\gamma_0 = -(B-1)/2 + (P_3-1)/2 + Q_3/2 - \theta/4 - [(B-1) + (\theta - \pi/2)/4]^2$$

and

$$u_0 = (1, -1)'.$$

The causality issue is not only interesting but it is essential to an understanding of the global dynamics of power systems with loads. We will carry the analysis of this example a little further. The algebraic equations (52b and c) must be satisfied for all admissible trajectories. They define a one-dimensional manifold of possible values in the three-dimensional space of variables (θ, ϕ, V). This manifold is called the configuration manifold. The state space consists of the points of the configuration manifold along with the one-dimensional tangent space associated with each point, i.e., the tangent bundle. Equation (52a) induces a vector field on the state space which is well defined at almost all points. The exceptional points are associated with points on the configuration manifold at which the matrix A becomes singular.

We will develop our example further to illustrate these points. It is convenient to change coordinates from (θ, ϕ, V) to (α, β, V) where

$$\alpha = (2\phi - \theta)/2, \qquad \beta = \theta/2 \tag{53}$$

and in reverse

$$\theta = 2\beta, \qquad \phi = \alpha + \beta. \tag{54}$$

In terms of the new variables, (52) becomes

$$\beta'' = -V\cos\alpha\sin\beta + \Delta P/2 \tag{55a}$$

$$0 = 2V\sin\alpha\cos\beta - P_3 \tag{55b}$$

$$0 = 2V\cos\alpha\cos\beta - (2-B)V^2 + Q_3. \tag{55c}$$

The configuration manifold is defined by (55b and c). It is easy to solve for V in terms of α

$$V = [(P_3 \operatorname{ctn}\alpha + Q_3)/(2-B)]^{1/2}. \tag{56}$$

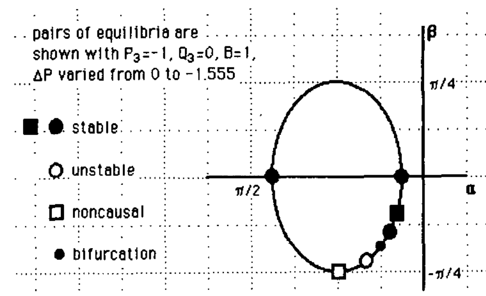

Fig. 2. Configuration manifold and equilibria.

Now, V can be eliminated from (55b) to yield a relation between α and β. The resulting configuration manifold is composed of an infinite number of disconnected and bounded components for almost all values of the parameters. Fig. 2 illustrates the principal component for $P_3 = -1$, $Q_3 = 0$, and $B = 1$. Equilibria are defined by setting the right-hand side of (55a) equal to zero. We consider the equilibrium point structure for various values of ΔP.

a) $\Delta P = 0$. Two equilibrium points exists on the principal component of the configuration manifold $(\alpha, \beta, V) = (-1.222, 0, .5333)$ and $(-.2618, 0, 1.932)$. It is easy to confirm that both equilibria are stable. As ΔP is decreased, the two equilibria move down along the manifold and remain stable until

b) $\Delta P = -1$. The two equilibria are now located at $(\alpha, \beta, V) = (-.7854, -.7854, 1)$ and $(-.2875, -.2877, 1.839)$. The lower voltage equilibrium is noncausal and the higher voltage equilibrium is stable. Further decreases in the value of ΔP results in both equilibria in the lower right quadrant of the configuration manifold; in this case, the lower voltage equilibrium is unstable and the higher is stable, until

c) $\Delta P = -1.555$. One equilibrium point exists and it is located at $(\alpha, \beta, V) = (-.4233, -.6109, 1.487)$. This equilibrium point is easily confirmed to be a bifurcation point and to be strictly causal. Any further decrease in ΔP results in the absence of any equilibrium points.

Consider once again the situation of a) above with $\Delta P = 0$. Now, suppose ΔP remains fixed and B is decreased. It can be shown that the principal component of the configuration manifold shrinks in size (with equilibria remaining at the two points at which $\beta = 0$) and reduces to a point as $B \to 0$. This is one view of the mechanism of voltage collapse associated with Case 2.

VI. Conclusions

In this paper, we have presented some results in the study of static stability of electric power systems as bifurcation phenomena. This work is a natural extension of that of Araposthatis, Sastry, and Varaiya [27] and is based on an expanded model, allowing transfer conductances and load models other than constant admittance. These enhancements are necessary to characterize power system voltage instabilities and allow us to give a precise definition of power system static stability. It has been shown that static bifurcation theory is a suitable framework for the definition, classification, and analysis of voltage collapse phenomena. Moreover, our experience strongly suggests that symbolic computation can make this theory a valuable practical tool.

The concept of regularity of an equilibrium point has been introduced and is an essential part of the definition of static stability. It has been shown that nonregular equilibria include bifurcation points of the reduced load flow equations and equilibria which are not strictly causal. We have shown how sensitivity factors often used to characterize voltage instabilities arise naturally from the basic concepts of bifurcation theory. A complete classification of static bifurcations associated with a null space of co-dimension one is available and is suitable for the analysis of power system static bifurcations generically characterized by a relatively small number of parameters (say, less than about six). It has been shown how the critical parameters can be identified using the Liapunov-Schmidt reduction followed by Taylor expansion of the reduced bifurcation equation. The ability to compute the functions $\gamma(\lambda)$ is potentially of great importance. When these functions are known, the critical physical parameters associated with a particular instability can be identified by sensitivity analysis, thus providing important information for system or control design.

The issue of causality becomes important when load buses are present, in which case the system motion takes place on an imbedded manifold. Although the studies reported herein deal with local phenomena, they indicate that a complete understanding of global power system dynamics will require a modern differential-geometric perspective which deals directly with motion on such a manifold. We have seen that some static instabilities may be associated with distortions of the configuration manifold under parameter variation which lead to loss of causality. This point of view raises many questions concerning even the most basic tools of power system analysis such as simulation techniques and direct methods of stability analysis.

Our analysis, based on static bifurcation of constrained differential equations, leads to loss of static stability by two mechanisms: divergence instability, or loss of causality. Other mechanisms related to local and global dynamic bifurcations are clearly possible and warrant further study. Furthermore, the model upon which our analysis is based can be extended in several ways which will undoubtedly reveal additional mechanisms of instability. The method of analysis described herein can be directly applied to static load characteristics other than constant power. The generator models can be generalized. In particular, automatic voltage regulators have not been considered here and can play an important role in some circumstances. Similarly, the dynamics of local voltage control devices (e.g., tap changing transformers, capacitor banks) may be important in certain instances.

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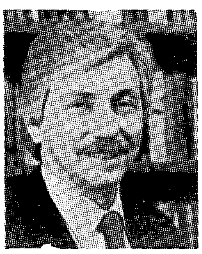

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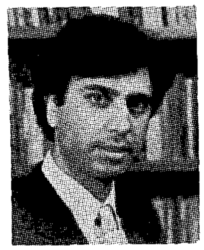

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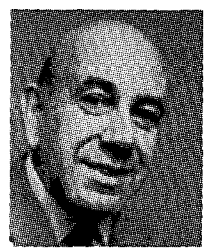

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